Joint Gaussian Graphical Model Series – V

sparse Gaussian Graphical Model estimators

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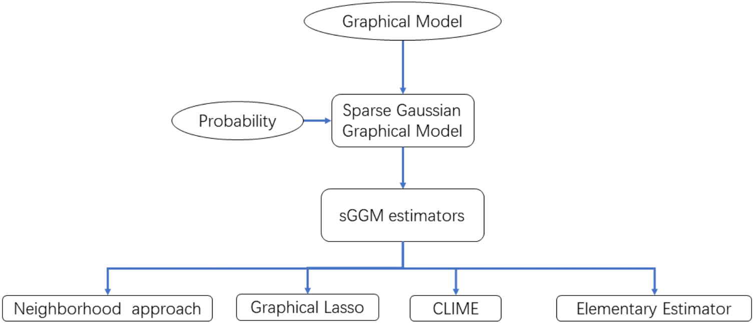
Road Map


Graphical Model
Probability
Sparse Gaussian Graphical Model
sGGM estimators
Neighborhood approach
Graphical Lasso
CLIME
Elementary Estimator

Outline

Notation

Notation

- Σ The covariance matrix.
- Ω The precision matrix.
- p The number of features.
- n The number of samples.

Review

Review from last talk

- Regularized M-estimator $\underset{\theta}{\operatorname{argmin}} \mathcal{L}(\theta) + \lambda_n \mathcal{R}(\theta)$
- a unified framework to analyze the statistical convergence rate for high-dimensional statistics
- Elementary Estimator

Review of Gaussian Graphical Model

Suppose the precision matrix $\Omega = \Sigma^{-1}$.

The log-likelihood of Ω equals to $\ln\det(\Omega) - \mathrm{tr}\left(\Omega\widehat{S}\right)$

In this talk, we will use this likelihood to derive several estimators of sparse Gaussian Graphical Model (sGGM)

Neighborhood Method

Neighborhood approach

If $X \sim N(0, \Sigma)$ and let $X_1 = X_j$.
$X_j | X_{\backslash j}\ N(\Sigma_{\backslash j,j}\Sigma^{-1}_{\backslash j,\backslash j}X_{\backslash j}, \Sigma_{jj} - \Sigma_{\backslash j,j}\Sigma^{-1}_{\backslash j,\backslash j}\Sigma_{\backslash j,j})$
Let $\alpha_j := \Sigma_{\backslash j,j}\Sigma^{-1}_{\backslash j,\backslash j}$ and $\sigma_j^2 := \Sigma_{jj} - \Sigma_{\backslash j,j}\Sigma^{-1}_{\backslash j,\backslash j}\Sigma_{\backslash j,j}$. We have that

$$X_j = \alpha_j^T X_{,\backslash j} + \epsilon_j \tag{3.1}$$

where $\epsilon_j \sim N(0, \sigma_j^2)$ is independent of $X_{,\backslash j}$.

Neighborhood approach with sparse assumption

By the sparse assumption, we estimate each α_j by a lasso estimator

$$\alpha_j = \operatorname*{argmin}_{\alpha_j} ||\alpha_j^T X_{,\setminus j} - X_j||_2^2 + \lambda ||\alpha_j||_1 \tag{3.2}$$

Review of Lasso solution

Lasso

$$\beta = \underset{\beta}{\mathrm{argmin}}\, ||\beta^T X - y||_2^2 + \lambda ||\beta||_1 \tag{3.3}$$

subgradient method

$$g(\beta;\lambda) = -2X^T(y - X\beta) + \lambda \mathrm{sgn}(\beta) \tag{3.4}$$

We see that the proximity operator is important because x^* is a minimizer to the problem $\min_{x \in \mathcal{H}} F(x) + R(x)$ if and only if $x^* = \text{prox}_{\gamma R}\left(x^* - \gamma \nabla F(x^*)\right)$, where $\gamma > 0$. γ is any positive real number.

Proximal gradient method

$$\left(\text{prox}_{\gamma R}(x)\right)_i = \begin{cases} x_i - \gamma, & x_i > \gamma \\ 0, & |x_i| \leq \gamma \\ x_i + \gamma, & x_i < -\gamma, \end{cases} \tag{3.5}$$

By using the fixed point method, you can obtain the estimation of β.

Graphical Lasso

Graphical Lasso[Friedman et al.(2008)Friedman, Hastie, and Tibshirani

We already have the log-likelihood as the loss function. Can we use it to obtain a similar estimator as Lasso?

$$\underset{\Omega}{\operatorname{argmin}} - \ln \det(\Omega) + \operatorname{tr}\left(\Omega \widehat{S}\right) + \lambda_n ||\Omega||_1 \tag{4.1}$$

Proximal gradient method to solve it

Let's do a practice in the white board.

Super Linear algorithm.

$\lim_{k\to\infty} \frac{|x_{k+1}-x^*|}{|x_k-x^*|} = 0.$

State of the art method: Big & QUIC[Hsieh et al.(2011)Hsieh, Sustik, Dhillon, and Ravikum

Parallelized Coordinate descent.

approximated quadratic algorithm.

$\lim_{k\to\infty} \frac{|x_{k+1}-x^*|}{|x_k-x^*|^2} < M$

CLIME

CLIME[Cai et al.(2011)Cai, Liu, and Luo]

CLIME

$$\underset{\Omega}{\operatorname{argmin}} ||\Omega||_1 \quad \text{, subject to: } ||\Sigma\Omega - I||_\infty \leq \lambda \tag{5.1}$$

Here $\lambda > 0$ is the tuning parameter.

By taking the first derivative of Eq. (4.1) and setting it equal to zero, the solution $\widehat{\Omega}_{glasso}$ also satisfies:

$$\widehat{\Omega}^{-1}_{glasso} - \widehat{\Sigma} = \lambda \widehat{Z} \tag{5.2}$$

where $\widehat{Z}$ is an element of the subdifferential $\partial ||\widehat{\Omega}_{glasso}||_1$.

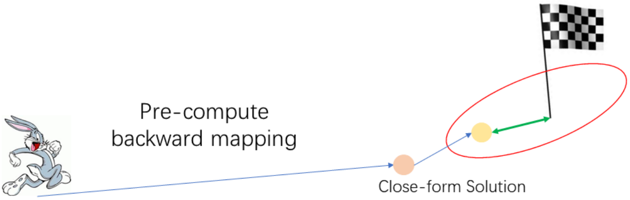
Column-wise estimator

$$\operatorname{argmin} ||\beta||_1 \quad \text{subject to} \quad ||\Sigma\beta - e_j||_\infty \leq \lambda$$

CLIME can be estimated column-by-column.

Elementary Estimator for Gaussian Graphical Model

Elementary Estimator



Elementary Estimator[Yang et al.(2014b)Yang, Lozano, and Ravikumar]

$$\underset{\theta}{\operatorname{argmin}} \mathcal{R}(\theta) \tag{6.1}$$

$$\text{Subject to: } \mathcal{R}^*(\theta - \mathcal{B}^*(\widehat{\phi})) \leq \lambda_n \tag{6.2}$$

Here $\mathcal{B}^*(\widehat{\phi})$ is a backward mapping for $\widehat{\phi}$.

Example: sparse linear regression[Yang et al.(2014a)Yang, Lozano, and Ravikumar]

$$\operatorname*{argmin}_{\theta} ||\theta||_1 \tag{6.3}$$

$$\text{Subject to: } ||\theta - (X^T X + \epsilon I)^{-1} X^T y||_\infty \leq \lambda_n \tag{6.4}$$

Elementary Estimator for sGGM

$$\begin{aligned} &\operatorname*{argmin}_{\Omega} |\Omega|_{1,off} \\ \text{subject to:} &|\Omega - [T_v(\Sigma)]^{-1}|_{\infty,off} \leq \lambda_n \end{aligned} \tag{6.5}$$

Summary

- We review most sGGM estimators.

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