Learning Kernels with Random Features

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Outline

1. Introduction
   - Motivation
   - Background
   - State-of-the-art
2. Proposed Approach
   - Work-flow
   - Formulation
   - Efficient solution
3. Evaluation
   - Learning a kernel
   - Feature Selection
   - Benchmark Datasets

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- **Weakness:** Poor choice of user defined kernel can lead to a useless model.
- **Goal:** Combine kernel learning with randomization.
- **Idea:** Exploit computational advantage of randomized features for supervised kernel learning.

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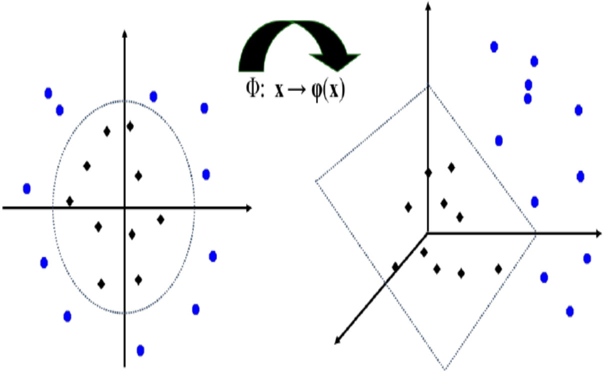
Kernel



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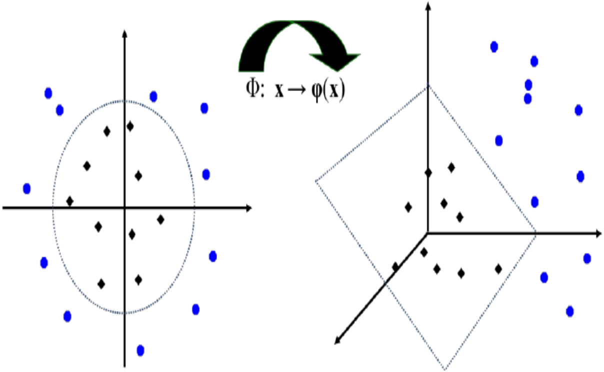

$$K(x_i, x_j) = < \phi(x_i), \phi(x_j) > \quad (1)$$

Random Features for Kernel [Rahimi and Recht, NIPS '07]

$$K(x, y) =< \phi(x), \phi(y) >= z(x)'z(y) \tag{2}$$

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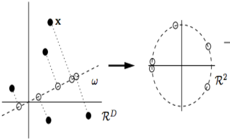

Kernel Name	$k(\Delta)$	$p(\omega)$
Gaussian	$e^{-\frac{\|\Delta\|_2^2}{2}}$	$(2\pi)^{-\frac{D}{2}} e^{-\frac{\|\omega\|_2^2}{2}}$
Laplacian	$e^{-\|\Delta\|_1}$	$\prod_d \frac{1}{\pi(1+\omega_d^2)}$
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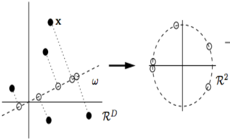

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Algorithm 1 Random Fourier Features.

Require: A positive definite shift-invariant kernel $k(\mathbf{x},\mathbf{y}) = k(\mathbf{x}-\mathbf{y})$.

Ensure: A randomized feature map $\mathbf{z}(\mathbf{x}) : \mathcal{R}^d \to \mathcal{R}^D$ so that $\mathbf{z}(\mathbf{x})'\mathbf{z}(\mathbf{y}) \approx k(\mathbf{x}-\mathbf{y})$.

Compute the Fourier transform p of the kernel k: $p(\omega) = \frac{1}{2\pi} \int e^{-j\omega'\delta} k(\delta)\, d\Delta$.

Draw D iid samples $\omega_1, \cdots, \omega_D \in \mathcal{R}^d$ from p and D iid samples $b_1, \dots, b_D \in \mathcal{R}$ from the uniform distribution on $[0, 2\pi]$.

Let $\mathbf{z}(\mathbf{x}) \equiv \sqrt{\frac{2}{D}} \left[\cos(\omega_1'\mathbf{x}+b_1) \cdots \cos(\omega_D'\mathbf{x}+b_D) \right]'$.

Outline

Kernel Learning

- Heuristic rules to combine kernels
- Optimize structured compositions of kernels w.r.t an alignment metric [Elaborate]
- Jointly optimize kernel composition with empirical risk

Outline

Work-flow

- Create randomized features
- Solve an optimization problem to select a subset
- Train a model with the optimized features
- Learn lower dimensional models than original random-feature approach

Outline

Formulation

- For binary classification: given n data points $(x^i, y^i) \in R^d \times \{-1, 1\}$. Let $\phi : R^d \times \mathcal{W} \to [-1, 1]$ and Q be a probability measure on a space $\mathcal{W}$, a kernel can be defined as:

$$K_Q(x, x') := \int \phi(x, w)\phi(x', w)dQ(w) \tag{3}$$

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- Given some base (user defined) distribution P_0, Consider collections $\mathcal{P} := \{Q : D_f(Q||P_0 \leq \rho)\}$, where $\rho > 0$ is a specified constant.

Formulation contd.

- Using randomized feature approach, approximate integral (3) as discrete sum over samples $W^i \sim P_0, i \in [N_w]$
- Approximate to $\mathcal{P} : \mathcal{P}_{N_w} := \{q : D_f(q||1/N_w) \leq \rho\}$
- Problem (4) becomes:

$$maximize_{q \in \mathcal{P}_{N_w}} \sum_{i,j} y^i y^j \sum_{m=1}^{N_w} q_m \phi(x^i, w^m) \phi(x^j, w^m) \tag{5}$$

Formulation contd.

- Given a solution $\hat{q}$, two ways to solve learning problem:
- Draw D samples $W^1, \ldots, W^d \sim \hat{q}$ defining features $\phi^i = [\phi(x^i, w^1) \ldots \phi(x^i, w^D)]^T$ and solve :

$$\hat{\theta} = argmin_{\theta}\{\sum_{i=1}^{n} c\left(\frac{1}{\sqrt{D}}\theta^T \phi^i, y^i\right) + r(\theta)\} \tag{6}$$

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- This is a two step approach

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- **Consistency**: Solution to problem (5) approaches a population optimum as data and random sampling increases.
- **Generalization**: Class of estimators used has strong performance guarantees.

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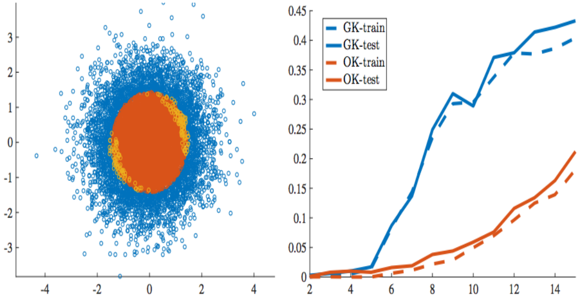

(a) Training data & optimized features for $d = 2$ (b) Error vs. d

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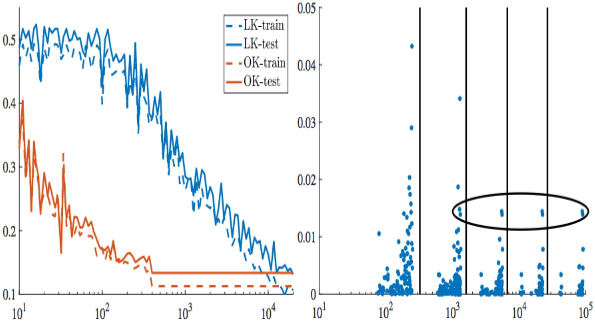

(a) Error vs. D (b) $\widehat{q}_i$ vs. i

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Benchmark Datasets

Table 1: Best test results over benchmark datasets

Dataset	n, n_{test}	d	Model	Our error (%), time(s)	Random error (%), time(s)
adult	32561, 16281	123	Logistic	15.54, 3.6	15.44, 43.1
reuters	23149, 781265	47236	Ridge	9.27, 0.8	9.36, 295.9
buzz	105530, 35177	77	Ridge	4.92, 2.0	4.58, 11.9

Summary

- Learn a kernel in a supervised manner using random features.
- Demonstrate consistency and generalization of the method.
- Attain competitive results on benchmark datasets with a fraction of training time.
- Future Direction
  - Usefulness of simple optimization methods on random features in speeding up traditionally expensive learning problems.